Indian Statistical Institute
B.Math. (Hons.) I Year
First Semester Exam, 2006-2007
Probability Theory I

Time: 3 hrs | Date: 01-12-06 | Max. Marks : 100

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Note: The paper carries 105 marks. Any score above 100 will be treated as 100.

1. A coin is tossed independently n times; let $0 < p < 1$ denote the probability of getting head in any individual toss. Let X denote the difference between the number of heads and the number of tails. (Note that X can take negative values also.)

   (i) Find the discrete density function of X.

   (ii) Find $E(X)$ and $\mathrm{Var}(X)$.

   (Hint: Find a relation between X and S_n, the number of heads.)

   [12+8]

2. Let X and Y be independent discrete random variables each having a geometric distribution with parameter $0 < p < 1$.

   (i) Show that $X + Y$ is a discrete random variable.

   (ii) Find the discrete density function of $X + Y$.

   (iii) For $k = 0, 1, 2, \ldots$ find the conditional distribution of Y given $X + Y = k$.

   [6+7+7]

3. Using Chebyshev's inequality show that

$$\lim_{n\to\infty} \sum_{j \leq n\theta} \frac{e^{-n} n^j}{j!} = \begin{cases} 0 , & \text{if} \quad \theta < 1 \\ 1 , & \text{if} \quad \theta > 1 \end{cases} .$$

   [15]

4. Let X be an absolutely continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{x} \frac{1}{(\log x)^2}, & \text{if} \quad x > e \\ 0, & \text{otherwise} \end{cases}$$

(i) Find $P(X > x)$ for $x > 0$.

(ii) Does X have finite expectation? Justify. (Hint: get an appropriate lower bound for $\int_e^x \frac{1}{\log t} dt$.)

[10+10]

5. Let X have $N(\mu,\ \sigma^2)$ distribution where $\mu \in I\!R,\ \sigma^2 > 0$. Let $Y = aX + b,\ a \neq 0$.

(i) Find the distribution of Y.

(ii) Can one find a transformation of the above type so that Y is standard normal?

[12+3]

6. Let X have the standard normal distribution.

(i) Find the moment generating function of X.

(ii) For $t \in I\!R$ define

$$f_t(x) = \frac{1}{m(t)} e^{tx} f(x), \quad x \in I\!R$$

where $f(\cdot), m(\cdot)$ are respectively the probability density function, moment generating function of X. Show that $f_t(\cdot)$ is the probability density function of some $N(\mu, \sigma^2)$ distribution.

[10+5]